E0 232 Probability and Statistics *Due:* Aug 21, 2012

Assignment 1

1. Let A, B and C be three events. Express each of the following events in terms of A, B and C using set operations (complementation, union, and intersection). In each case, also draw the corresponding Venn diagrams.
   (a) Exactly one of the events A or B occurs.
   (b) At least two of the events A, B, and C occur.
   (c) At most one of A, B, and C occurs.
   (d) A and C occur but not B.
   (e) Either A and C both occur, or B occurs.
   (f) None of A, B, C occur.

2. There are two jars, one containing 10 white coupons numbered 1–10, and the other containing 25 red coupons numbered 1–25. Your friend picks a coupon randomly from each jar; assume a uniform probability law under which each pair of white and red coupons is equally likely to be picked.
   (a) What is the probability that the white coupon picked has a number divisible by 3?
   (b) What is the probability that the white coupon picked has a number divisible by 3 or the red coupon picked has 1, 2, or 17?
   (c) What is the probability that the white coupon picked has a number divisible by 3 and the red coupon picked has 1, 2, or 17?
   (d) What is the probability that the white coupon picked has a number divisible by 3 and the red coupon has 1, 2, or 17, or the white coupon picked has an even number?

3. A new student in the CSA department at IISc must take 4 courses in the first term. There are a total of 20 courses to choose from, divided into 3 pools: pools A and B contain 6 courses each, numbered A1–A6 and B1–B6, while pool C contains 8 courses, numbered C1–C8. A student Anusha decides to choose courses randomly as follows: she first selects a pool uniformly at random from the 3 pools. She then picks 2 courses uniformly at random from the selected pool, and one course each uniformly at random from the other two pools.
   (a) What is the size of the sample space in this experiment?
   (b) What is the probability that course A1 is selected?
   (c) What is the probability that courses C3 and C7 are both selected?
   (d) What is the probability that 2 courses are selected from pool B, and neither A3 nor C8 is selected?
   (e) Assume A1 is selected. What is the probability that 2 courses from pool B were selected?

4. Let A, B, C, and D be four events, such that A, B, and C are disjoint with $P(A) = 0.6$, $P(B) = 0.3$, and $P(C) = 0.1$, and $P(D|A) = 0.3$, $P(D|B) = 0.9$, and $P(D|C) = 0.7$. Find the following probabilities:
   (a) $P(B \cap D)$.
   (b) $P(B \cap D \cap A)$.
   (c) $P((B \cap D) \cup (C \cap D))$.
   (d) $P(A|D)$.
   (e) $P(D|A \cup B)$.

5. A fair coin is tossed repeatedly. What is the probability that on the nth toss:
   (a) a head appears for the first time?
   (b) the numbers of heads and tails so far (up to and including the nth toss) are equal?
   (c) exactly 3 heads have appeared so far?
   (d) at least 2 heads have appeared so far?

6. Ashish has 6 pairs of socks: 2 white, 2 blue, and 2 black. After washing the 12 socks, he randomly pairs them into 6 pairs, so that all pairings are equally likely (he does not distinguish between left and right socks; all look identical to him).
   (a) What is the probability that all 6 pairs he has formed have socks of a uniform color?
   (b) What is the probability that exactly 2 of the 6 pairs have socks of a uniform color, and both of those pairs are of the same color?
   (c) What is the probability that none of the 6 pairs has socks of a uniform color?
   (d) On each of the next 6 days, Ashish will wear one of the 6 pairs he has formed. On the first four days, you see Ashish wearing black-blue, black-white, black-black, and blue-white. What is the probability that he will be wearing socks of a uniform color the next day?

7. Let $A_1, \ldots, A_n$ be n events. Prove the following inclusion-exclusion formula:

$$P\left(\cup_{i=1}^n A_i\right) = \sum_i P(A_i) - \sum_{i_1<i_2} P(A_{i_1} \cap A_{i_2}) + \sum_{i_1<i_2<i_3} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \ldots + (-1)^{n-1} P\left(\cap_{i=1}^n A_i\right)$$

   Hint: Use induction.

8. Let $A_1, A_2, \ldots$ be an infinite sequence of events such that $A_n \subseteq A_{n+1}$ for all n. Let $A = \cup_{n=1}^\infty A_n$. Show the following continuity property of probabilities:

$$P(A) = \lim_{n\to\infty} P(A_n)$$

   Hint: Express the event A as a union of countably many disjoint events.