E0 232 Probability and Statistics *Due:* Sept 4, 2012

Assignment 2: Answers

1. (a) $p_X(k) = P(X = k) = \begin{cases} \frac{1}{3}, & \text{if } k = 1, \\ \frac{4}{15}, & \text{if } k = 2, \\ \frac{1}{5}, & \text{if } k = 3, \\ \frac{2}{15}, & \text{if } k = 4, \\ \frac{1}{15}, & \text{if } k = 5, \\ 0 & \text{otherwise.} \end{cases}$

   (b) $F_X(k) = P(X \leq k) = \begin{cases} 0, & \text{if } k < 1, \\ \frac{1}{3}, & \text{if } 1 \leq k < 2, \\ \frac{9}{15}, & \text{if } 2 \leq k < 3, \\ \frac{12}{15}, & \text{if } 3 \leq k < 4, \\ \frac{14}{15}, & \text{if } 4 \leq k < 5, \\ 1, & \text{if } k \geq 5, \end{cases}$

   (c) $\mathbf{E}[X] = \frac{7}{3} = 2.333$
   $Var[X] = \frac{14}{9} = 1.556$

   (d) $\frac{1}{5} = 0.2$

   (e) i. $p_Y(k) = P(Y = k) = \begin{cases} (\frac{4}{6})^{k-1}(\frac{2}{6}) & \text{if } k \in \{1, 2, \ldots\} \\ 0 & \text{otherwise.} \end{cases}$

   ii. $F_Y(k) = P(Y \leq k) = \begin{cases} 1 - (\frac{4}{6})^{\lfloor k \rfloor} & \text{if } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$
   where $\lfloor k \rfloor$ is the greatest integer less or equal to k

   iii. $\mathbf{E}[Y] = 3$
   $Var[Y] = 6$

   iv. $P(Y > 3) = \frac{8}{27}$

   (f) i. $W = 20 \times \mathbf{1}(X > 3)$
   $p_W(k) = P(W = k) = \begin{cases} P(X \leq 3) = \frac{12}{15} & \text{if } k = 0 \\ P(X > 3) = \frac{3}{15} & \text{if } k = 20 \\ 0 & \text{otherwise} \end{cases}$

   ii. $F_W(k) = P(W \leq k) = \begin{cases} \frac{12}{15} & \text{if } 0 \leq k < 20 \\ 1 & \text{if } k \geq 20 \end{cases}$

   iii. $\mathbf{E}[W] = 4$
   $Var[W] = \frac{4}{5} \times (0-4)^2 + \frac{1}{5} \times (20-4)^2 = 64$

2. $p_X(k) = P(X = k) = \begin{cases} \binom{2n-1-k}{n-1} \times \frac{1}{2^{2n-1-k}} & \text{if } k \in \{1, 2, \ldots, n\} \\ 0 & \text{otherwise.} \end{cases}$

3. (a) $p_{Y|X}(y|x) = \begin{cases} \binom{x}{y} q^y (1-q)^{x-y} & \text{if } y \in \{0, 1, \ldots, x\} \\ 0 & \text{otherwise} \end{cases}$

   (b) $p_{X,Y}(x, y) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} \times \binom{x}{y} q^y (1-q)^{x-y} & \text{if } x \in \{0, 1, \ldots, n\}, y \in \{0, 1, \ldots, x\} \\ 0 & \text{otherwise.} \end{cases}$

(c) $p_Y(y) = \begin{cases} \binom{n}{y}(pq)^y(1-pq)^{n-y} & \text{if } y \in \{0,1,\ldots,n\} \\ 0 & \text{otherwise.} \end{cases}$

(d) $\frac{p(1-q)}{1-pq}$

(e) $X - Y|Y = y \sim Binomial\left(n-y, \frac{p(1-q)}{1-pq}\right)$

$p_{X|Y}(x|y) = \begin{cases} \frac{\binom{n}{x}p^x(1-p)^{n-x} \times \binom{x}{y}q^y(1-q)^{x-y}}{\binom{n}{y}(pq)^y(1-pq)^{n-y}} & \text{if } y \in \{0,1,\ldots,n\}, x \in \{y, y+1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$.

(f) $\mathbf{E}[X|Y=2] = (n-2)\frac{p(1-p)}{1-pq} + 2$

4. (a) $p_X(k) = P(X=k) = \begin{cases} \binom{n}{l}(p)^l(1-p)^{n-l} & \text{if } l = \frac{n+k}{2} \in \{0,1,\ldots,n\} \\ 0 & \text{otherwise} \end{cases}$

(b) $E[X] = 2np - n$
$\text{var}[X] = 4np(1-p)$

(c) $p_{X|A}(k) = P(X=k|A) = \begin{cases} \frac{P(X=k)}{\sum_{l=-3}^{3} P(X=l)} & \text{if } k \in \{-3,-2,-1,0,1,2,3\} \\ 0 & \text{otherwise.} \end{cases}$

5. (a) $\mathbf{E}[R] = 7 \times 0.5 = 3.5$.
$Var[R] = 1.75$.

(b) $\mathbf{E}[T] = 2 \times \mathbf{E}[R] + 15 = 22$.
$Var[T] = 2^2 \times Var[R] = 7$.

6. (a) $c = \frac{3}{7}$

(b) $F_X(x) = \begin{cases} 0 & \text{if } x < 1, \\ \frac{3(x^3-1)}{21} & \text{if } 1 \le x \le 2, \\ 1 & \text{if } 2 < x, \end{cases}$

(c) $P(\frac{5}{4} < X < \frac{7}{4}) = \frac{218}{7 \times 64}$

(d) $E[X] = \frac{45}{28}$
$\text{var}(X) = \frac{93}{35} - \frac{45^2}{28^2}$

7. (a) $c = 1$

(b) $f_X(x) = \begin{cases} 2(x-1) & \text{if} 1 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$

(c) $P\left(\frac{5}{4} < X < \frac{7}{4}\right) = \frac{1}{2}$.

(d) $\mathbf{E}[X] = \frac{5}{3}$
$Var[X] = \frac{1}{36}$

8. See notes on probabilistic method sent by email.