E0 232 Probability and Statistics *Due:* Sep 4, 2012

Assignment 2

1. You have 6 writing pens in your bag, of which 4 are old and have stopped working, and 2 are new. In order to write your assignment, you need a working pen. You draw a pen uniformly at random from your bag; if it doesn't work, you throw it into the dustbin and repeat. Let X be the number of draws needed to retrieve a working pen.

   (a) Write out clearly the probability mass function of X. Draw a plot of the function.

   (b) Write out clearly the cumulative distribution function of X. Draw a plot of the function.

   (c) Find the expectation and variance of X.

   (d) Find $P(X > 3)$.

   (e) Now suppose that when you draw a pen that doesn't work, you forget to throw it into the dustbin; instead, you put it back into your bag, and again draw a pen uniformly at random. Let Y be the number of draws needed to retrieve a working pen in this case. Repeat parts (a)-(d) for Y.

   (f) Suppose again that you follow the first procedure, where you draw pens uniformly at random and throw pens that don't work into the dustbin. However, now, if after 3 tries you don't find a working pen, then you go to the bookstore and buy a new pen for Rs 20. Let W be the amount of money you spend (in Rs). Write W as a function of X, and repeat parts (a)-(c) for W.

2. There are two jars, each containing n balls. Each day, you choose one of the jars uniformly at random, and remove one ball from that jar. What is the PMF of the number of balls remaining when one of the jars becomes empty?

3. Each day, you play a game of badminton with your friend with probability p, independently of whether you played on any other day. Each time you play, the probability you win is q, independently of everything else. Let X be the number of days on which you play in a fixed period of n days ($n \geq 2$), and Y be the number of days on which you win.

   (a) Find the conditional PMF $p_{Y|X}(y|x)$.

   (b) Find the joint PMF $p_{X,Y}(x, y)$.

   (c) Find the marginal PMF $p_Y(y)$.
   Hint: While you can start with the joint PMF obtained in part (b), this might be messy. Try thinking directly about how Y is generated.

   (d) What is the probability that you played on any particular day given that you did not win any game that day?

   (e) Find the conditional PMF $p_{X|Y}(x|y)$.

   (f) Find $E[X|Y = 2]$.

4. Consider a particle that lives on the infinite number line, starting at position 0. At each time step, it moves by $+1$ with probability p and -1 with probability $1 - p$, independently of its moves at previous time steps. Let X denotes its position after n time steps.

   (a) Find the PMF of X.

   (b) Find the expectation and variance of X.

   (c) Let A be the event that $X \in \{-3, -2, -1, 0, 1, 2, 3\}$. Find the conditional PMF $p_{X|A}(x)$ and the conditional expectation $E[X|A]$.

5. Ayesha passes through seven traffic lights on her way to work. Each light is equally likely to be red or green, independent of the others.
   (a) Find the PMF, mean, and variance of the number of red lights that Ayesha encounters.
   (b) If there are no red lights, it takes Ayesha 15 minutes to commute to work. Each red light adds a delay of 2 minutes. Find the mean and variance of her commute time (in minutes).

6. Let X be a random variable with probability density function given by
$$f_X(x) = \begin{cases} c\,x^2 & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise,} \end{cases}$$
   where c is a constant.
   (a) Find the value of c.
   (b) Write out clearly the cumulative distribution function of X.
   (c) Find $P\Big(\frac{5}{4} < X < \frac{7}{4}\Big)$.
   (d) Find the expected value and variance of X.

7. Let X be continuous a random variable with cumulative distribution function given by
$$F_X(x) = \begin{cases} c\,(x-1)^2 & \text{if } 1 \le x \le 2 \\ 1 & \text{if } x > 2 \\ 0 & \text{otherwise,} \end{cases}$$
   where c is a constant.
   (a) Find the value of c.
   (b) Write out clearly the probability density function of X.
   (c) Find $P\Big(\frac{5}{4} < X < \frac{7}{4}\Big)$.
   (d) Find the expected value and variance of X.

8. Let $G = (V, E)$ be the complete (undirected) graph on n vertices $V = \{1, \ldots, n\}$ (so that $|E| = \binom{n}{2}$). A *2-coloring* of the edges of G is an assignment of each edge of G to one of 2 colors, say red and blue. Show that for any $k < n$ such that $2^{\binom{k}{2}-1} > \binom{n}{k}$, there exists a 2-coloring of the edges of G such that every complete subgraph of G on k vertices contains both red and blue edges.
   Hint: Construct a probability distribution over the set of possible 2-colorings, and show that under the given condition on k, the probability that a randomly selected 2-coloring contains a complete size-k subgraph that has only red or only blue edges is strictly smaller than 1.