E0 232 Probability and Statistics *Due:* Sep 18, 2012

Assignment 3

1. Let $X \sim \mathcal{N}(\mu, \sigma^2)$ be a normal random variable with PDF given by
$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad \forall x \in \mathbb{R}.$$
   (a) Show that f_X satisfies the normalization property, i.e. show that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.
   Hint: Use the fact that $\left(\int_{-\infty}^{\infty} f_X(x)\,dx\right)^2 = \left(\int_{-\infty}^{\infty} f_X(x)\,dx\right)\left(\int_{-\infty}^{\infty} f_X(y)\,dy\right)$ *and transform to polar coordinates.*
   (b) Show that $\mathrm{var}(X) = \sigma^2$.
   Hint: Use the definition of variance; to evaluate the integral, use a change of variable to $y = \frac{x-\mu}{\sigma}$.

2. Let $X \sim \mathcal{N}(1, 4)$.
   (a) Find $P(X > 1)$.
   (b) Find $P(2 < X < 3)$.
   (c) Let $Y = 2X + 5$. Find $P(Y > 2)$.

3. Suppose you install a new bulb in a room. The lifetime T_1 of the bulb (in days) is a random variable distributed as $T_1 \sim \mathrm{Exp}(\lambda)$ for some $\lambda > 0$. You leave the bulb on, and after t days, you go and check whether the bulb is still working. If not, then you replace the bulb with a new bulb whose lifetime T_2 (in days) is also distributed as $T_2 \sim \mathrm{Exp}(\lambda)$, independent of T_1. Again, you leave the bulb on. What is the expected amount of time (in days) during which the room is illuminated?

4. Let X and Y be two random variables with joint PDF given by
$$f_{X,Y}(x,y) = \begin{cases} c\,x^2 y & \text{if } 0 \le x \le y \le 2 \\ 0 & \text{otherwise,} \end{cases}$$
   where c is a constant.
   (a) Find the value of c.
   (b) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.
   (c) Find the conditional PDFs $f_{Y|X}(y|x)$ and $f_{X|Y}(x|y)$.
   (d) Find $E[X]$, $E[Y]$, $\mathrm{var}(X)$, $\mathrm{var}(Y)$, and $\mathrm{cov}(X,Y)$.
   (e) Find $P((X,Y) \in [1,2] \times [1,2])$.

5. Alice and Bob are scheduled to meet at a given time. Each of them is late by a random amount of time; Alice's delay X (in minutes) is distributed as $X \sim \mathrm{Exp}(\lambda)$; Bob's delay (in minutes) is distributed as $Y \sim \mathrm{Exp}(\mu)$, independent of X. Let T be the delay (in minutes) of the first person to arrive. Find the PDF of T.

6. Let X_1, $X_2 \sim \mathrm{Exp}(\lambda)$ be two independent random variables. Let $X = X_1 + X_2$. Find the PDF of X.

7. Let X, Y be independent random variables, where X is uniformly distributed over $[-2, 2]$, and Y is generated by flipping a fair coin and drawing a number uniformly at random from $[-3, -1]$ if the coin lands heads and uniformly from $[1, 3]$ if it lands tails. Let $W = X^2$ and let $Z = X + Y$.
   (a) Find the PDF of W.
   (b) Find $\mathrm{cov}(W, X)$.
   (c) Find the PDF of Z.
   (d) Find $\mathrm{cov}(Z, X)$.