E0 232 Probability and Statistics *Due:* Oct 25, 2012

Assignment 5

1. An old telephone exchange initiates telephone calls in millisecond intervals, and can initiate at most one call during any such millisecond interval (if more than one call is received during the interval, only the first call received is initiated; other calls are dropped). In each millisecond interval, there is a probability $\frac{1}{10}$ of a call being initiated, independent of other intervals.

   (a) What is the expected number of calls initiated in a 1-minute period (whose start coincides with the start of a millisecond interval as above)?

   (b) Find (exactly or approximately) the probability that more than 6200 calls are initiated in a 1-minute period as above.

   (c) Let X be the time in milliseconds until the 100th call is initiated. Find the PMF of X.

   (d) Let Y be the time in seconds until the 100th call is initiated. Find the PMF of Y.

   (e) Each call received is a local call with probability $\frac{3}{5}$ and a long-distance call with probability $\frac{2}{5}$, independently of all other calls. What is the probability that on a given day, the 50th local call initiated takes place within the first second?

2. Buses arrive at the IISc bus stop according to a Poisson process with arrival rate 2/hour. Autorickshaws arrive according to an independent Poisson process with arrival rate 10/hour. Hema, Anant, and Ishan reach the bus stop at the same time. Hema takes the first vehicle (bus or autorickshaw) that arrives. Anant is in no rush; he decides to wait for 30 minutes, and then catches the first bus that arrives after that. Ishan decides he will take the 11th vehicle (bus or autorickshaw) to arrive.

   (a) What is Hema's expected waiting time?

   (b) What is Anant's expected waiting time?

   (c) What is Ishan's expected waiting time?

   (d) What is the probability that Anant sees no buses during his 30-minute wait?

   (e) What is the probability that Anant sees 4 buses and 2 autorickshaws during his 30-minute wait?

   (f) What is the probability that of the 10 vehicles Ishan skips, 6 are autorickhsaws and 4 are buses?

   (g) What is the probability that Ishan leaves before Anant?

   (h) What is the probability that Anant and Ishan end up on the same bus?

   (i) What is the probability that Hema and Anant end up on the same bus?

3. Consider a Bernoulli process with success probability p in each trial.

   (a) Let A be the event that there is exactly 1 success in the first 10 trials. Find the conditional PMF of the first arrival time, Y_1, given A, $p_{Y_1|A}(y)$. What can you conclude from this?

   (b) Let B be the event that there are exactly 2 successes in the first 10 trials. Find the joint PMF of the first two arrival times, Y_1 and Y_2, conditioned on B, $p_{Y_1,Y_2|B}(y_1, y_2)$. What can you conclude?

4. Consider a Poisson process with arrival rate λ per time unit.

   (a) Let A be the event that there is exactly 1 arrival in the first 10 time units. Find the conditional PDF of the first arrival time, Y_1, given A, $f_{Y_1|A}(y)$. What can you conclude from this?

   (b) Let B be the event that there are exactly 2 arrivals in the first 10 time units. Find the joint PDF of the first two arrival times, Y_1 and Y_2, conditioned on B, $f_{Y_1,Y_2|B}(y_1, y_2)$. What can you conclude?

5. Consider a Markov chain on three states $\{1, 2, 3\}$ with transition probability matrix

$$\begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0.3 & 0.7 & 0 \\ 0.1 & 0.8 & 0.1 \end{bmatrix}$$

Let X_n denote the state at time n.

(a) Find $P(X_1 = 2, X_2 = 3, X_3 = 1 | X_0 = 3)$.

(b) Suppose X_0 is selected uniformly at random from $\{1, 2, 3\}$. Find $P(X_1 = 2, X_2 = 3, X_3 = 1)$.

(c) Find the 2-step transition probability matrix, i.e. the matrix whose (i, j)-th entry is the probability $P(X_2 = j | X_0 = i)$.

(d) Find the 3-step transition probability matrix, i.e. the matrix whose (i, j)-th entry is the probability $P(X_3 = j | X_0 = i)$.

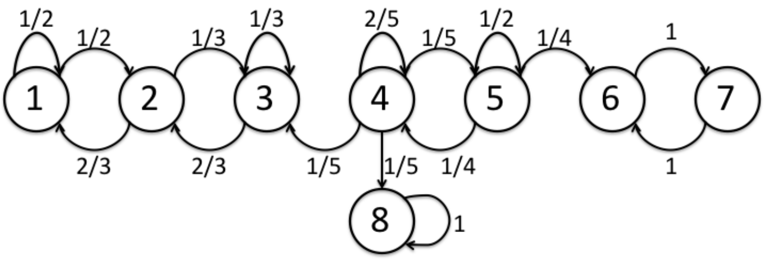
(e) Does this Markov chain have a steady-state distribution? If so, find this distribution.

6. Let $Y_1, Y_2, \ldots$ be a sequence of iid random variables with PMF

$$p_{Y_i}(y) = \begin{cases} 0.4 & \text{if } y = 1 \\ 0.3 & \text{if } y = 2 \\ 0.2 & \text{if } y = 3 \\ 0.1 & \text{if } y = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Let $X_0 = 1$ and let $X_n = \max(Y_1, \ldots, Y_n)$ for $n \geq 1$. Show that $X_0, X_1, \ldots$ forms a Markov chain. Find the probability transition matrix and draw the corresponding probability transition graph. Does this chain have a steady-state distribution? If so, find this distribution.

7. Consider a Markov chain with the following transition probability graph:



(a) Label the states as recurrent or transient. Identify the recurrent classes.

(b) Starting in state 4, what is the expected number of time steps until the chain leaves state 4?

(c) Starting in state 4, what is the probability that the chain reaches state 1 at some point?

(d) Starting in state 3, what is the expected number of time steps until state 1 is reached?

(e) Starting in state 4, what is the expected number of time steps until a recurrent state is reached?

(f) Starting in state 3, does the chain have a steady-state distribution? If so, find this distribution.

(g) Starting in state 7, does the chain have a steady-state distribution? If so, find this distribution.

8. A professor gives tests that are hard, medium, or easy. If she gives a hard test, her next test will be either medium or easy, with equal probability. However, if she gives a medium or easy test, there is a 0.6 probability that the next test will be hard, and a 0.2 probability for each of the other two levels of difficulty. Construct an appropriate Markov chain for this process. What is the steady-state probability of a hard test?